MIDTERM: INTRODUCTION TO ALGEBRAIC GEOMETRY

Date: 21^{st} **February 2025**
The Total points is **112** and the maximum you can score is **100** points.

A **ring** would mean a **commutative ring with identity**.

(1) (15 points) For $m, c \in \mathbb{C}$ let R be the ring $\mathbb{C}[x,y]/I$ where I is the ideal $(y - x^2, y - mx + c)$. Find all values of m, c for which R is a reduced ring.

(2) (15 points) Let $R = \mathbb{C}[x, y, (x+1)^{-1}, (x-1)^{-1}]/(y - x^2 + 1)$ and $R' = S^{-1}\mathbb{C}[x,y]/(y - x^2 + 1)$ where $S = \{1, \bar{y}, \bar{y}^2, \ldots\}$. Show that R and R' are isomorphic.

(3) (6+6=12 points) Prove or disprove.
   (a) Let X be an algebraic subset of $\mathbb{A}^n_{\mathbb{C}}$ then $Z(I(X)) = X$.
   (b) Let J be an ideal of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ then $I(Z(J)) = J$.

(4) (5+15=20 points) Define irreducible subsets and components of an algebraic set. Compute the irreducible components of the affine algebraic subset of $\mathbb{A}^2_{\mathbb{C}}$ defined by the polynomial $f(x,y) = (x^2+y^2)(x^2+y^2+1)(x^4-y^4)$. What are the minimal primes of the ideal (f) in $\mathbb{C}[x,y]$? Compute $\sqrt{(f)}$.

(5) (10+10+10=30 points) Let k be an algebraically closed field. Define affine varieties, its coordinate ring and morphism of varieties. Prove or disprove the following.
   (a) Let $X = Z(xyz-1)$ in $\mathbb{A}^3_k$ with coordinates x, y, z. There is a surjective morphism of affine varieties from $X \to \mathbb{A}^1_k$.
   (b) Let $f : X \to Y$ be a morphism of varieties over an algebraically closed field k induced from the inclusion of k-algebras $k[Y] \subset k[X]$. The morphism f is surjective.

(6) (5+15=20 points) State going up theorem. Find an example of a ring extension $A \subset B$ such that the induced map from $\operatorname{Spec}(B)$ to $\operatorname{Spec}(A)$ is surjective but going up property for the extension fails.